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AUTHOR(S):

Goto, Ryushi

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ON ELLIPTIC FIBRATIONS AND HYPER-KÄHLER STRUCTURES

RYUSHI GOTO

Osaka University

§0. INTRODUCTION

In this paper we shall discuss the class of Ricci-flat manifolds called hyper-Kähler manifolds. A hyper-Kähler manifold is, by definition, a Riemannian manifold equipped with three complex structures I, J, K satisfying the quaternionic relations, with respect to all of which the metric is Kählerian. The standard example of hyper-Kähler manifold is the vector space over the quaternion. The well known compact hyper-Kähler manifolds are $K3$ surfaces and even dimensional complex tori. A class of hyper-Kähler manifolds arises as moduli spaces of certain geometric structures, such as the moduli spaces of instantons on S^4 , monopoles on \mathbb{R}^3 or Higgs bundles on Riemannian surfaces.

Eguchi-Hanson firstly discovered an interesting example of noncompact complete hyper-Kähler manifold which is diffeomorphic to the holomorphic cotangent bundle of \mathbb{CP}^1 . Its $4m$ dimensional generalization was obtained by Calabi. He showed that the holomorphic cotangent bundle of \mathbb{CP}^m has a hyper-Kähler structure. Gibbons-Hawking constructed hyper-Kähler structures on all minimal resolutions of rational double points of type A_k . They are called hyper-Kähler 4 manifolds of type A_k . Kronheimer-Nakajima constructed $4m$ dimensional noncompact complete hyper-Kähler manifolds which generalize hyper-Kähler 4 manifolds of type A_k . In [G-1], the author has studied geometrical and topological properties of these hyper-Kähler $4m$ manifolds of type A_k .

The first purpose of this paper is to construct new families of noncompact complete hyper-Kähler 4 manifolds. In [G-2], we shall give the $4m$ dimensional generalization. The following table (i) will show the significance of our new manifolds.

Hyper-Kähler manifolds of type A

	4 dim	$4m$ dim ($m > 1$)
A_1	$T^*\mathbb{CP}^1$ (by Eguchi-Hanson)	$T^*\mathbb{CP}^m$ (by Calabi)
A_k	Hyper-Kähler 4 manifolds of type A_k (by Gibbons-Hawking)	$4m$ dimensional hyper- Kähler manifolds of type A_k (in [KN], [G-1])
A_∞	Hyper-Kähler manifolds of type A_∞ (constructed in this paper)	$4m$ dimensional hyper- Kähler manifolds of type A_∞ (constructed in [G-2])
A_∞^+	Hyper-Kähler manifolds of type A_∞^+ (in [AKL])	$4m$ dimensional hyper- Kähler manifolds of type A_∞^+ (constructed in [G-2])

Table (i)

As shown in Table (i), we obtain two families of type A_∞ and of type A_∞^+ . In 4 dimensional case, hyper-Kähler manifolds of type A_∞^+ were constructed by Anderson-Kronheimer-Lebrun [AKL]. But our construction is different from their construction. Our construction of hyper-Kähler 4-manifolds of type A_∞ is well understood by the comparison with Kronheimer's construction of hyper-Kähler 4-manifolds of type A_k . This comparison will be well explained by the following table (ii).

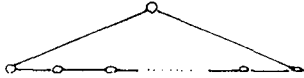
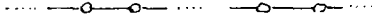
Construction of hyper-Kähler 4 manifolds of type A_k by Kronheimer	Construction of hyper-Kähler 4 manifolds of type A_∞ in this paper
cyclic group Γ_k of order $k + 1$ in $SU(2)$	maximal torus S^1 in $SU(2)$
extended Dynkin diagram of type A_k 	extended Dynkin diagram of type A_∞ 
regular representation R of Γ_k	regular representation $L^2(S^1)$ of S^1
module over \mathbb{H} $M = (\text{End}(R) \otimes_{\mathbb{C}} \mathbb{H})^{\Gamma_k}$	Hilbert manifold $\hat{M} \subset (\text{Hom}(L^2(S^1)) \otimes_{\mathbb{C}} \mathbb{H})^{S^1}$
Lie group $G = U(R)^{\Gamma_k}$	Hilbert Lie group $G \subset U(L^2(S^1))^{S^1}$
hyper-Kähler moment map μ	map μ given by (0-1)
The hyper-Kähler quotient $\mu^{-1}(\zeta)/G$	Our new quotient $\mu^{-1}(\zeta)/G$

Table (ii)

Let us explain Kronheimer's construction briefly. His construction has relied on the hyper-Kähler quotient construction. He has used the regular representation R of Γ_k to construct the \mathbb{H} module $M = (\text{Hom}(R) \otimes_{\mathbb{C}} \mathbb{H})^{\Gamma_k}$ and the Lie group $G = U(R)^{\Gamma_k}$, where M is a set of Γ_k -invariant elements of $\text{Hom}(R) \otimes_{\mathbb{C}} \mathbb{H}$ and G is the set of Γ_k -invariant unitary map of R . The Lie group G acts on the \mathbb{H} -module M preserving its hyper-Kähler structure. Then we have the hyper-Kähler moment map $\mu: M \rightarrow \text{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathfrak{g}^*$, where $\text{Im}\mathbb{H}$ is the imaginary part of the quaternion and \mathfrak{g}^* the dual space of the Lie algebra of G . For generic $\zeta \in \text{Im}\mathbb{H} \otimes \mathfrak{g}^*$, we obtain a hyper-Kähler manifold $\mu^{-1}(\zeta)/G$. Then Kronheimer has showed that this hyper-Kähler manifold $\mu^{-1}(\zeta)/G$ coincides with hyper-Kähler manifold of type A_k . Moreover he has showed that this construction can be described in terms of the extended Dynkin diagram. (Note that Kronheimer has constructed hyper-Kähler metrics on all minimal resolutions of rational double points.)

In our case of hyper-Kähler manifolds of type A_∞ , we shall use the regular representation $L^2(S^1)$ of the maximal torus S^1 of $SU(2)$ to construct the Hilbert manifold \hat{M} and the Hilbert Lie group \hat{G} . The Hilbert manifold \hat{M} is a subset of $(\text{Hom}(L^2(S^1)) \otimes_{\mathbb{C}} \mathbb{H})^{S^1}$ of S^1 -invariant elements of $\text{Hom}(L^2(S^1)) \otimes_{\mathbb{C}} \mathbb{H}$, where

$\text{Hom}(L^2(S^1))$ is the set of operators of $L^2(S^1)$ whose domains are dense. The Hilbert Lie group \hat{G} is a subgroup of S^1 -invariant unitary operators $U(L^2(S^1))^{S^1}$. An element of \hat{M} can be regarded as a \mathbb{H} valued operator $\alpha + \beta j$, where α, β are operators of $L^2(S^1)$. We shall define the map $\hat{\mu}$ from \hat{M} to $\text{Im}\mathbb{H} \otimes \mathfrak{g}^*$ by

$$(0-1) \quad \begin{cases} \mu_I(\alpha + \beta j) &= ([\alpha, \alpha^*] + [\beta, \beta^*]), \\ \mu_{\mathbb{C}}(\alpha + \beta j) &= -2\sqrt{-1} [\alpha, \beta]. \end{cases}$$

where $\mu = \mu_I + \mu_{\mathbb{C}}j$. Then a hyper-Kähler manifold of type A_∞ will be constructed as a quotient space $\mu^{-1}(\zeta)/G$ for generic $\zeta \in \text{Im}\mathbb{H} \otimes \mathfrak{g}^*$. Moreover we shall show that our construction can be described in terms of the extended Dynkin diagram of type A_∞ in table (ii). For the construction of hyper-Kähler manifolds of type A_∞^+ , we shall use the representation space $L^{2+}(S^1)$ of L^2 functions on the circle S^1 whose negative Fourier coefficients vanish. Then we follow the same procedure as in type A_∞ . This construction of type A_∞^+ corresponds to the following extended Dynkin diagram of type A_∞^+ .



where each vertex has the weight number 1.

The second purpose of this paper is to discuss the relation between hyper-Kähler manifolds of type A_∞ and elliptic fibrations. Let X_ζ be the hyper-Kähler manifold of type A_∞ . When we consider a certain subfamily of hyper-Kähler manifolds of type A_∞ , we can choose a special complex structure I and a holomorphic symplectic form $\omega_{\mathbb{C}}$ on each X_ζ in this subfamily. Then there exist three kind of actions on X_ζ . At first \mathbb{C}^* acts on X_ζ preserving I and $\omega_{\mathbb{C}}$. We denote by Φ the holomorphic moment map on X_ζ for the action of \mathbb{C}^* . (We shall describe this moment map Φ explicitly.) Secondly there exists the holomorphic action of an additive group \mathbb{Z} on X_ζ . Since the moment map Φ is invariant under the action of \mathbb{Z} , we have a map $\tilde{\Phi}$ from the quotient space $X_\zeta/b\mathbb{Z}$ to \mathbb{C} for any positive integer b . Denote by Δ the disk $\{ t \in \mathbb{C} \mid |t| < 1 \}$. When we restrict the map $\tilde{\Phi}$ to the inverse image $\tilde{\Phi}^{-1}(\Delta)$, we have the map $\hat{\Phi} : \tilde{\Phi}^{-1}(\Delta) \rightarrow \Delta$. Then we obtain the following Theorem.

Main theorem. $\hat{\Phi} : \tilde{\Phi}^{-1}(\Delta) \rightarrow \Delta$ is biholomorphic to the fibre space of elliptic curves of type I_b .

$$\begin{array}{ccccc} X_\zeta & \longrightarrow & X_\zeta/b\mathbb{Z} & \longleftarrow & \tilde{\Phi}^{-1}(\Delta) \\ & \searrow & \downarrow \Phi & & \downarrow \Phi \\ & & \mathbb{C} & \longleftarrow & \Delta \end{array}$$

Remark. All fibre space of elliptic curves over the disk were classified by a celebrated theorem of Kodaira. Our notation is the same as in his paper [KK].

Colollary. *The hyper-Kähler 4 manifold of type A_∞ is the universal cover of the fibre space of elliptic curves of type I_b .*

Finally we shall show that there exists the holomorphic involution σ on X_ζ . By the action of σ , we can obtain the fibre space of elliptic curves of type I_b^* . In the final theorem of this paper, we shall discuss hyper-Kähler manifolds of type D_∞ . We shall show that hyper-Kähler manifolds of type D_∞ can be constructed by the regular representation of the normalizer of S^1 in $Sp(1)$ as in type A_∞ [G-4].

§1. PRELIMINARY RESULTS

In this section, we shall give a brief review of the hyper-Kähler quotient construction and Kronheimer's result.

Definition 1-1. *A hyper-Kähler structure on a Riemannian manifold (X, g) consists of three almost complex structures (I, J, K) which satisfy following conditions*

(1)

$$g(u, v) = g(Iu, Iv) = g(Ju, Jv) = g(Ku, Kv).$$

(2)

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K \in \text{End}(TX).$$

(3) Let ∇ be a Levi-Civita connection of (X, g) . Then

$$\nabla I = \nabla J = \nabla K = 0.$$

A *hyper-Kähler manifold* is a Riemannian manifold with a hyper-Kähler structure. Especially the module over the quaternion is a standard example of a hyper-Kähler manifold. Next we define a hyper-Kähler moment map. Let (X, g, I, J, K) be a hyper-Kähler manifold. We assume that a Lie group G acts on X so as to preserve the hyper-Kähler structure of X . Each element $\xi \in \mathfrak{g}$ of the Lie algebra of G defines a vector field $\hat{\xi}$ on X by the action of G .

Definition 1-2. *A hyper-Kähler moment map for the action of G on M is a map $\mu = i\mu_I + j\mu_J + k\mu_K: M \rightarrow \text{Im}\mathbb{H} \otimes \mathfrak{g}^*$ which satisfies*

$$\mu_{I_\alpha}(gx) = \text{Ad}_g^*(\mu_{I_\alpha})(x), \quad x \in M, g \in G, \alpha = 1, 2, 3$$

$$\langle \xi, d\mu_{I_\alpha} \rangle = i(\hat{\xi})\omega_{I_\alpha}, \quad \xi \in \mathfrak{g}, \alpha = 1, 2, 3$$

where $(I_1, I_2, I_3) = (I, J, K)$, \mathfrak{g}^* the dual space of \mathfrak{g} , $\text{Ad}_g^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ the coadjoint map, $\langle \cdot, \cdot \rangle$ the dual pairing between \mathfrak{g} and \mathfrak{g}^* , and $i(\hat{\xi})$ the interior product.

Under the assumption that M is simply connected, a hyper-Kähler moment map always exists and is unique up to addition of a constant $\zeta \in \text{Im}\mathbb{H} \otimes Z \subset \text{Im}\mathbb{H} \otimes \mathfrak{g}^*$, where Z is the set of G -invariant elements of \mathfrak{g}^* . The set $\mu^{-1}(\zeta) \in M$ is invariant under G -action for any $\zeta \in \text{Im}\mathbb{H} \otimes Z$. After choosing $\zeta \in \text{Im}\mathbb{H} \otimes \mathfrak{g}^*$, one defines the *hyper-Kähler quotient* as

$$X_\zeta = \mu^{-1}(\zeta)/G.$$

We are now ready to state the hyper-Kähler quotient construction.

Fact 1-3[H-K-L-R]. Suppose that G acts freely on $\mu^{-1}(\zeta)$, then the hyper-Kähler quotient $\mu^{-1}(\zeta)/G = X_\zeta$ is a hyper-Kähler manifold. Moreover if G is compact and M is complete, then X_ζ is a complete hyper-Kähler manifold.

Kronheimer constructed the family of hyper-Kähler structures on minimal resolutions of rational double points. Let Γ be a finite subgroup of $SU(2)$. There is the natural action of Γ on the quaternion \mathbb{H} by the identification $SU(2) \cong Sp(1)$. Denote by R the regular representation with the invariant Hermitian metric. Then $\text{End}(R) \otimes_{\mathbb{C}} \mathbb{H}$ is regarded as a module over the quaternion on which Γ acts preserving its hyper-Kähler structure. Let $M := (\text{End}(R) \otimes_{\mathbb{C}} \mathbb{H})^\Gamma$ be the set of invariant elements of $\text{End}(R) \otimes_{\mathbb{C}} \mathbb{H}$ under the action of Γ . Denote by G the group of unitary maps of R which are invariant under the adjoint action of Γ . Then M is the module over the quaternion on which the compact Lie group G acts so as to preserve the hyper-Kähler structure.

When we apply hyper-Kähler quotient construction on M and G , we obtain the following Theorem by Kronheimer.

Theorem 1-4. Let M, G be as before. Let μ be the hyper-Kähler moment map from M to $\text{Im}\mathbb{H} \otimes \mathfrak{g}^*$. For generic $\zeta \in \text{Im} \otimes Z$, the hyper-Kähler manifold $\mu^{-1}(\zeta)/G$ is diffeomorphic to the minimal resolution of \mathbb{C}^2/Γ .

§2. HYPER-KÄHLER MANIFOLDS OF TYPE A_∞

Let V_n be a irreducible representation of the circle group S^1 which is generated by the function $e^{in\theta}$ on S^1 . By the basis $e^{in\theta}$ each V_n may be regarded as the one dimensional complex vector space \mathbb{C} . Consider the following diagram with all edges doubled up and assigned orientations both ways :

$$\cdots \quad \begin{array}{c} \xrightarrow{\alpha_{n-2}} \\ \xleftarrow{\beta_{n-2}} \end{array} V_{n-1} \quad \begin{array}{c} \xrightarrow{\alpha_{n-1}} \\ \xleftarrow{\beta_{n-1}} \end{array} V_n \quad \begin{array}{c} \xrightarrow{\alpha_n} \\ \xleftarrow{\beta_n} \end{array} V_{n+1} \quad \begin{array}{c} \xrightarrow{\alpha_{n+1}} \\ \xleftarrow{\beta_{n+1}} \end{array} \cdots$$

where each arrow implies a homomorphism between irreducible representations. Denote by (α_n, β_n) an element of $\text{Hom}(V_n, V_{n+1}) \oplus \text{Hom}(V_{n+1}, V_n)$. Consider the infinite dimensional module

$$H := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V_n, V_{n+1}) \oplus \text{Hom}(V_{n+1}, V_n).$$

We define the Hilbert space M by

$$M := \left\{ (\alpha_n, \beta_n)_{n \in \mathbb{Z}} \in H \quad \mid \quad \sum_{n \in \mathbb{Z}} |\alpha_n|^2 + |\beta_n|^2 < \infty \right\}$$

Since M is a vector space over \mathbb{C} , we have the almost complex structure I on M by the mutiplication of i . An almost complex structure J is defined by

$$J(\alpha_n, \beta_n)_{n \in \mathbb{Z}} := (\beta_n^*, -\alpha_n^*)_{n \in \mathbb{Z}}.$$

When we set $K = IJ$, then M has a hyper-Kähler structure. Define an element $\Lambda = (\Lambda_n)_{n \in \mathbb{Z}}$ by

$$\Lambda_n = \begin{cases} (ni, 0) & \text{if } n \geq 0, \\ (0, ni) & \text{if } n < 0. \end{cases}$$

Definition 2-1. We define the Hilbert manifold \hat{M} by

$$\hat{M} := \Lambda + M \subset H$$

An element of \hat{M} may be written as $(\alpha_n, \beta_n)_{n \in \mathbb{Z}}$ where $(\alpha_n, \beta_n) = \Lambda_n + (x_n, y_n)$ for $(x_n, y_n)_{n \in \mathbb{Z}} \in M$. Define the Hilbert space \mathfrak{g} by

$$\mathfrak{g} := \left\{ (\xi_n)_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} u(V_n) \quad \mid \quad \sum_{n \in \mathbb{Z}} (1 + n^2) |\xi_n - \xi_{n+1}| < \infty, \quad \lim_{|n| \rightarrow \infty} \xi_n = 0 \right\}.$$

Definition 2-2. Define the Hilbert Lie group G by

$$G := \left\{ (e^{\xi_n})_{n \in \mathbb{Z}} \in \prod_{n \in \mathbb{Z}} U(V_n) \mid (\xi_n)_{n \in \mathbb{Z}} \in \mathfrak{g} \right\}.$$

The Lie algebra of the Hilbert Lie group G is the Hilbert space \mathfrak{g} . Since G is the subgroup of $\prod_{n \in \mathbb{Z}} U(V_n)$, we have the action of G on $H := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V_n, V_{n+1}) \oplus \text{Hom}(V_{n+1}, V_n)$. Then the Hilbert manifold \hat{M} is invariant under the action of G . Hence we have the action of G on \hat{M} . It is clear that the G acts on \hat{M} preserving the hyper-Kähler structure. Then there exists the hyper-Kähler moment map μ on M for the action of G .

$$\mu : M \longrightarrow \text{Im} \mathbb{H} \otimes \mathfrak{g}^*.$$

The map μ is described by

$$\begin{aligned} \langle \mu^I(q), \xi^I \rangle &= \sum_{n \in \mathbb{Z}} \langle (\alpha_n^* \alpha_n - \beta_n \beta_n^* - \alpha_{n-1} \alpha_{n-1}^* + \beta_{n-1}^* \beta_{n-1}), \xi_n^I \rangle \\ &\quad - \langle C_n^I(\Lambda), \xi_n^I \rangle \\ \langle \mu^C(q), \xi^C \rangle &= \sum_{n \in \mathbb{Z}} 2i \langle \beta_n \alpha_n - \alpha_{n-1} \beta_{n-1}, \xi_n^C \rangle - \langle C_n^C(\Lambda), \xi_n^C \rangle, \end{aligned}$$

where $q = (\alpha_n, \beta_n)_{n \in \mathbb{Z}} \in \hat{M}$, $\xi = (i\xi_n^I + j\xi_n^J + k\xi_n^K)_{n \in \mathbb{Z}} \in \text{Im} \mathbb{H} \otimes \mathfrak{g}$, $\xi_n^C = \xi_n^J + i\xi_n^K$ and $C_n(\Lambda) := C_n^I(\Lambda) + C_n^C(\Lambda)$ is a constant which does not depend on $q \in \hat{M}$. We consider the following element $\hat{e}^n \in \text{Im} \mathbb{H} \otimes \mathfrak{g}$

$$\hat{e}_m^n = \begin{cases} i(i + j + k) & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

Definition 2-3. An element $\zeta \in \text{Im} \mathbb{H} \otimes \mathfrak{g}^*$ is said to be nondegenerate if $\sum_{i=n}^m \zeta_i \neq 0 \in \text{Im} \mathbb{H}$ for all $n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}$, where $\zeta_i := \langle \zeta, \hat{e}^n \rangle - \langle C_n(\Lambda), \hat{e}_n^n \rangle$.

Form the definition of μ , an inverse image $\mu^{-1}(\zeta)$ is invariant under the action of G . So we have the quotient space $X_\zeta := \mu^{-1}(\zeta)/G$ for $\zeta \in \text{Im} \mathbb{H} \otimes \mathfrak{g}^*$.

Theorem 2-4. The quotient space $X_\zeta := \mu^{-1}(\zeta)/G$ is a noncompact complete hyper-Kähler 4 manifold for any nondegenerate element $\zeta \in \text{Im} \mathbb{H} \otimes \mathfrak{g}^*$.

It is natural that X_ζ in Theorem 2-4 is called a hyper-Kähler manifold of type A_∞ by the following Theorem.

Theorem 2-5. Let X_ζ be a hyper-Kähler manifold of type A_∞ . Then there exist submanifolds $L_n \in X_\zeta, n \in \mathbb{Z}$ such that

- (1) each L_n is homeomorphic to the sphere S^2
- (2) the inclusion $\bigcup_{n \in \mathbb{Z}} L_n \subset X_\zeta$ is a deformation retract.
- (3) each intersection number is given by

$$L_{n_1} \cdot L_{n_2} = \begin{cases} -2 & \text{if } n_1 = n_2, \\ 1 & \text{if } |n_1 - n_2| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

The intersection form of $H(X_\zeta, \mathbb{Z})$ can be interpreted as (-1) times of Cartan matrix of type A_∞ .

Sketch of a proof of Theorem 2-4. For any nondegenerate element ζ , we see that $\mu^{-1}(\zeta)$ is a submanifold of \hat{M} by using an implicit function theorem. A key point of a proof is an existence of a slice on $\mu^{-1}(\zeta)$ for the action of G . Denote by S_q a slice on $q \in \mu^{-1}(\zeta)$. Then we have an orthogonal decomposition

$$T_q \mu^{-1}(\zeta) = T_q S_q + T_q G(q),$$

where $T_q G(q)$ is the tangent space of G -orbit through q . By the action of G on \hat{M} , each element $\xi \in \mathfrak{g}$ defines a vector field V_ξ on \hat{M} . So we define the map $d_q : \mathfrak{g} \rightarrow T_q \hat{M}$ by

$$d_q(\xi) := V_\xi(q) \in T_q \hat{M}.$$

Consider the following diagram

$$0 \longrightarrow \hat{\mathfrak{g}} \xrightarrow{d_q} T_q \hat{M} \xrightarrow{d\hat{\mu}_q} T_{\hat{\mu}(q)} \hat{N} \longrightarrow 0,$$

where $d\mu_q$ is the differential of the map μ at q . Let d_q^* be the adjoint operator of d_q . Then the decomposition $T_q \hat{\mu}^{-1}(\zeta) = T_q S_q + T_q \hat{G}_q$ implies that

$$T_q S_q \cong \text{Ker} d_q^* \cap \text{Ker} d\mu_q.$$

We define the map $D_q := d_q^* + d\mu_q : T_q \hat{M} \rightarrow \mathfrak{g} \oplus (\text{Im} \mathbb{H} \otimes \mathfrak{g}^*) \cong \mathbb{H} \otimes \mathfrak{g}^*$, where the image of the map d_q is in the real part of $\mathbb{H} \otimes \mathfrak{g}^*$ and \mathfrak{g} is identified with \mathfrak{g}^* by the metric. Then the map D_q satisfies the followings,

- (1) $D_q : T_q \hat{M} \rightarrow \mathbb{H} \otimes \mathfrak{g}^*$ is a linear operator between Hilbert spaces over \mathbb{H} ,
- (2) D_q is a Fredholm operator whose index is equal to 4 for all $q \in \mu^{-1}(\zeta)$,
- (3) $\text{Coker} D_q = \{0\}$.

We see that

$$T_q S_q = \text{Ker} D_q.$$

Hence $\dim_{\mathbb{R}} T_q S_q = \text{ind} D_q = 4$. Since D_q is a linear operator over \mathbb{H} , $\text{Ker} D_q$ is a vector space over \mathbb{H} . This implies that $T_q S_q$ has a hyper-Kähler structure. Let π denote a natural projection from $\mu^{-1}(\zeta)$ to $X_\zeta := \mu^{-1}(\zeta)/G$. Then each tangent space $T_x X_\zeta$ may be considered as $T_q S_q$ for $\pi(q) = x$. Hence X_ζ has an almost hyper-Kähler structure. Finally we can prove that this almost hyper-Kähler structure defines a hyper-Kähler structure.

§3. HOLOMORPHIC DESCRIPTION OF HYPER-KÄHLER MANIFOLDS OF TYPE A_∞

We use the same notation as in section 2. In this section, we choose $\zeta \in \mathfrak{ig} \subset \text{Im}\mathbb{H} \otimes \mathfrak{g}^*$, i.e., $\zeta^J, \zeta^K = 0$.

Proposition 3-1. *Let X_ζ be a hyper-Kähler manifold of type A_∞ and let L_n be submanifolds of X_ζ , $n \in \mathbb{Z}$. Then there exists a complex structure I on X_ζ such that each L_n is a complex submanifold with respect to I .*

This proposition implies that X_ζ has an infinite chain of rational curves. Since $\zeta^\mathbb{C} := \zeta^J + \zeta^K i = 0$, we have an inclusion $\mu^{-1}(\zeta) \hookrightarrow \mu_\mathbb{C}^{-1}(0)$, where $\mu_\mathbb{C} := \mu_J + \mu_K i$. By the explicit description of the hyper-Kähler moment map μ , we see that

$$\mu_\mathbb{C}^{-1}(0) = \left\{ (\alpha_n, \beta_n)_{n \in \mathbb{Z}} \in \hat{M} \mid \beta_n \alpha_n = \alpha_{n-1} \beta_{n-1}, \quad \forall n \in \mathbb{Z} \right\}.$$

We consider the following open subset $\mu_\mathbb{C}^{-1}(0)_+ \subset \mu_\mathbb{C}^{-1}(0)$,

$$\mu_\mathbb{C}^{-1}(0)_+ := \left\{ (\alpha_n, \beta_n)_{n \in \mathbb{Z}} \in \mu_\mathbb{C}^{-1}(0) \mid |\alpha_n|^2 + |\beta_n|^2 \neq 0, \quad \forall n \in \mathbb{Z} \right\}$$

For simplicity, we assume that $\zeta_i > 0$ for all $i \in \mathbb{Z}$. Then we have a natural inclusion

$$\mu^{-1}(\zeta) \hookrightarrow \mu_\mathbb{C}^{-1}(0)_+.$$

Let $G^\mathbb{C}$ denote the complexification of G . Then we have the map

$$\iota : X_\zeta \rightarrow \mu_\mathbb{C}^{-1}(0)_+/G^\mathbb{C}.$$

Theorem 3-2. $\mu_\mathbb{C}^{-1}(0)_+/G$ is a complex surface with a holomorphic symplectic form $\tilde{\omega}_\mathbb{C}$. The map $\iota : X_\zeta \rightarrow \mu_\mathbb{C}^{-1}(0)_+/G^\mathbb{C}$ is biholomorphic with respect to the complex structure I on X_ζ . Moreover $\iota^*(\tilde{\omega}_\mathbb{C}) = \omega_\mathbb{C}$.

§4. ELLIPTIC FIBRATIONS AND HYPER-KÄHLER MANIFOLDS OF TYPE A_∞ AND D_∞

Let (X_ζ, I) be a pair of a hyper-Kähler manifold of type A_∞ and a complex structure in Proposition 3-1. Then we have three kinds of actions on X_ζ .

- (1) the action of \mathbb{C}^* ,
- (2) the action of an additive group \mathbb{Z} ,
- (3) the involution σ .

At first we shall define the action of \mathbb{C}^* on X_ζ . Choose an element $(\alpha_n, \beta_n)_{n \in \mathbb{Z}} \in \mu_\mathbb{C}^{-1}(0)_+$. Denote by $[\alpha_n, \beta_n]_{n \in \mathbb{Z}}$ the equivalent class of $(\alpha_n, \beta_n)_{n \in \mathbb{Z}}$ in X_ζ . Then the action of \mathbb{C}^* is defined by

$$\phi : \mathbb{C}^* \times X_\zeta \longrightarrow X_\zeta$$

$$\phi(\lambda, [\alpha_n, \beta_n]_{n \in \mathbb{Z}})_n := \begin{cases} (\lambda \alpha_0, \lambda^{-1} \beta_0) & \text{if } n = 0 \\ (\alpha_n, \beta_n) & \text{if } n \neq 0. \end{cases}$$

It is clear that this definition is well defined. Since the action of \mathbb{C}^* on X_ζ is preserving the holomorphic symplectic form $\omega_\mathbb{C}$, we have the holomorphic moment map Φ on X_ζ for the action of \mathbb{C}^* .

$$\Phi : X_\zeta \longrightarrow \mathbb{C},$$

where \mathbb{C} is considered as the dual space of the Lie algebra of \mathbb{C} . The map Φ is explicitly described by

$$\Phi([\alpha_n, \beta]_{n \in \mathbb{Z}}) = \alpha_0 \beta_0.$$

Secondly we shall define the action of \mathbb{Z} on X_ζ . Consider the following map

$$f : \mu_\mathbb{C}^{-1}(0)_+ \rightarrow V$$

$$f((\alpha_n, \beta_n)_{n \in \mathbb{Z}}) := \begin{cases} (-in^{-1}\alpha_n, ni\beta_n) & \text{if } n > 0, \\ (\alpha_0, \beta_0) & \text{if } n = 0, \\ (ni\alpha_n, -in^{-1}\beta_n) & \text{if } n < 0, \end{cases}$$

where $V = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V_n, V_{n+1}) \oplus \text{Hom}(V_{n+1}, V_n)$. Let Y denote the image of the map f . Denote by \hat{X}_ζ the quotient space $Y/G^\mathbb{C}$. Then X_ζ may be considered as \hat{X}_ζ .

$$X_\zeta \cong \hat{X}_\zeta.$$

We define the map ψ by

$$\psi : \hat{X}_\zeta \longrightarrow \hat{X}_\zeta,$$

$$\psi([\hat{\alpha}_n, \hat{\beta}_n]_{n \in \mathbb{Z}}) := [\hat{\alpha}_{n-1}, \hat{\beta}_{n-1}]_{n \in \mathbb{Z}}.$$

Notice that $(\hat{\alpha}_{n-1}, \hat{\beta}_{n-1})_{n \in \mathbb{Z}}$ is an element of Y . The map ψ is well explained by the following diagram:

$$\begin{array}{ccccccc} \cdots & \begin{array}{c} \xrightarrow{\alpha_{n-2}} \\ \xleftarrow{\beta_{n-2}} \end{array} & V_{n-1} & \begin{array}{c} \xrightarrow{\alpha_{n-1}} \\ \xleftarrow{\beta_{n-1}} \end{array} & V_n & \begin{array}{c} \xrightarrow{\alpha_n} \\ \xleftarrow{\beta_n} \end{array} & V_{n+1} & \begin{array}{c} \xrightarrow{\alpha_{n+1}} \\ \xleftarrow{\beta_{n+1}} \end{array} & \cdots \\ & & & \Downarrow \psi & & & & & \\ \cdots & \begin{array}{c} \xrightarrow{\alpha_{n-3}} \\ \xleftarrow{\beta_{n-3}} \end{array} & V_{n-1} & \begin{array}{c} \xrightarrow{\alpha_{n-2}} \\ \xleftarrow{\beta_{n-2}} \end{array} & V_n & \begin{array}{c} \xrightarrow{\alpha_{n-1}} \\ \xleftarrow{\beta_{n-1}} \end{array} & V_{n+1} & \begin{array}{c} \xrightarrow{\alpha_n} \\ \xleftarrow{\beta_n} \end{array} & \cdots \end{array}$$

The action of \mathbb{Z} on \hat{X}_ζ is defined by

$$n \longrightarrow \psi^n \in \text{Aut}(\hat{X}_\zeta),$$

where $n \in \mathbb{Z}$. By the identification $X_\zeta \cong \hat{X}_\zeta$, we have the action of \mathbb{C}^* on X_ζ . Note that $\hat{\beta}_n \hat{\alpha}_n = \hat{\alpha}_{n-1} \hat{\beta}_{n-1}$ for all $n \in \mathbb{Z}$ where $(\hat{\alpha}_n, \hat{\beta}_n)_{n \in \mathbb{Z}} \in Y$. This implies that the holomorphic moment map $\tilde{\Phi}$ is invariant under the action of \mathbb{Z} . Hence we have the map $\tilde{\Phi} : X_\zeta / b\mathbb{Z} \rightarrow \mathbb{C}$ for any positive integer b .

$$\tilde{\Phi} : X_\zeta \longrightarrow X_\zeta / b\mathbb{Z}$$

Set $\Delta := \{ t \in \mathbb{C} \mid |t| < 1 \}$. Then we shall show that $\tilde{\Phi}^{-1}(\Delta) \rightarrow \Delta$ is biholomorphic to the fibre space of elliptic curves of type I_b .

Proof of Main theorem. When $t = \hat{\alpha}_0 \hat{\beta}_0 \neq 0$, we can define an invariant function x by

$$x := \left(\prod_{n \geq 0} \hat{\alpha}_n \right) \left(\prod_{n < 0} \hat{\beta}_n \right)^{-1},$$

where each infinite product converges absolutely. By a simple calculation, we see that (t, x) defines a local coordinate around a generic fibre $\Phi^{-1}(t)$ of X_ζ for $t \neq 0$. Moreover any general fibre is written as $\Phi^{-1}(t) \cong \mathbb{C}^* = \{ (t, x) \mid x \in \mathbb{C}^* \}$. The action of \mathbb{Z} can be described as the following,

$$\psi((t, x)) = (t, t^{-1}x).$$

This implies that each general fibre $\Phi^{-1}(t)$ is an elliptic curve $\mathbb{C}/(\mathbb{Z}\sqrt{-1} + \mathbb{Z}\log t)$. In order to determine a special fibre $\Phi^{-1}(0)$, we describe the infinite chain of rational curves in Theorem 3-1. Define \hat{L}_n by

$$\hat{L}_n := \left\{ [\hat{\alpha}_n, \hat{\beta}_n]_{n \in \mathbb{Z}} \in \hat{X}_\zeta \mid \hat{\alpha}_i = 0 (i < n), \quad \hat{\beta}_i = 0 (i \geq 0) \right\}.$$

Each \hat{L}_n is well explained by the following diagram:

$$\cdots \xleftarrow{\beta_{n-2}} V_{n-1} \xleftarrow{\beta_{n-1}} V_n \xrightarrow{\alpha_n} V_{n+1} \xrightarrow{\alpha_{n+1}} \cdots$$

Then we have a holomorphic map $\hat{L}_n \rightarrow \mathbb{CP}^1$ by

$$[\hat{\alpha}_n, \hat{\beta}_n]_{n \in \mathbb{Z}} \longrightarrow \left[\prod_{i \geq n} \hat{\alpha}_i, \prod_{i < n} \hat{\beta}_i \right] \in \mathbb{CP}^1.$$

We can see that this map is bijective. Hence the infinite chain of rational curves may be considered as $\hat{L}_n, n \in \mathbb{Z}$. By definition of ψ , we have

$$\psi(\hat{L}_n) = \hat{L}_{n+1}.$$

Moreover we see that $\Phi^{-1}(0) = \bigcup_{n \in \mathbb{Z}} \hat{L}_n$. This implies that a special fibre $\tilde{\Phi}^{-1}(0)$ is a circle of rational curves. Hence we can conclude that $\tilde{\Phi}^{-1}(\Delta) \rightarrow \Delta$ is biholomorphic to the fibre space of elliptic curves of type I_b . \square

Finally we shall define an involution σ on X_ζ . Consider the following map $\tilde{\sigma} : V \rightarrow V$ defined by,

$$\tilde{\sigma}((\alpha_n, \beta_n)_{n \in \mathbb{Z}})_n := \begin{cases} (-\beta_{-n-1}, \alpha_{-n-1}) & \text{if } n > 0, \\ (\beta_{-n-1}, -\alpha_{-n-1}) & \text{if } n \leq 0. \end{cases}$$

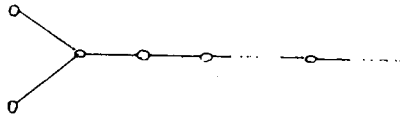
This map $\tilde{\sigma}$ is well understood by the following diagram:

$$\begin{array}{ccccccc} \cdots & \begin{array}{c} \xrightarrow{\alpha_{n-2}} \\ \xleftarrow{\beta_{n-2}} \end{array} & V_{-1} & \begin{array}{c} \xrightarrow{\alpha_{n-1}} \\ \xleftarrow{\beta_{n-1}} \end{array} & V_0 & \begin{array}{c} \xrightarrow{\alpha_n} \\ \xleftarrow{\beta_n} \end{array} & V_1 & \begin{array}{c} \xrightarrow{\alpha_{n+1}} \\ \xleftarrow{\beta_{n+1}} \end{array} & \cdots \\ & & & \Downarrow \tilde{\sigma} & & & & & \\ \cdots & \begin{array}{c} \xrightarrow{\beta_1} \\ \xleftarrow{-\alpha_1} \end{array} & V_{-1} & \begin{array}{c} \xrightarrow{\beta_0} \\ \xleftarrow{-\alpha_0} \end{array} & V_0 & \begin{array}{c} \xrightarrow{-\beta_{-1}} \\ \xleftarrow{\alpha_{-1}} \end{array} & V_1 & \begin{array}{c} \xrightarrow{-\beta_{-2}} \\ \xleftarrow{\alpha_{-2}} \end{array} & \cdots \end{array}$$

It is clear that $\tilde{\sigma}$ defines an involution σ on X_ζ . By using this involution σ , we obtain the fibre space of elliptic curves of type I_b^* . It must be noted that σ is a hyper-Kähler isometry. Hence we have a hyper-Kähler orbifold X_ζ/σ . This orbifold X_ζ/σ has two rational double points of type A_1 .

Theorem 4-1. (hyper-Kähler manifolds of type D_∞). *Let \tilde{X} be a minimal resolution of X_ζ/σ . Then \tilde{X} has a complete hyper-Kähler structure.*

Remark 4-2. A hyper-Kähler manifold \tilde{X} has a infinite sequence of rational curves. The dual graph of these rational curves coincides with the following Dynkin diagram of type D_∞ :



Remark 4-3. Let D_∞ denote by the normalizer of the maximal torus of $Sp(1)$. When we consider the Hilbert space $L^2(D_\infty)$, we can construct a family of hyper-Kähler manifolds by the hyper-Kähler quotient method. Then we can see that X_ζ/σ and \tilde{X} can be obtained as these hyper-Kähler quotient spaces which correspond to $L^2(D_\infty)$.

Hence it is natural that \tilde{X} of Theorem 4.1 is considered as the hyper-Kähler manifold of type D_∞ .

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